Math 254-2 Exam 0 Solutions

1. Carefully state the definition of "linear function". Give two examples.

   A linear function, on one or more variables, multiplies each variable by some constant, and adds the results together. Many examples are possible: $f(x, y) = 3x + 7y$, $g(x, y, z) = 8x + 0y + 2z$, $f(x) = 0$.

2. Carefully state the definition of "dimension". Give two examples.

   The dimension of a vector space is the number of elements in a basis of that vector space. Many examples are possible: $\mathbb{R}^2$ has basis $\{(1, 0), (0, 1)\}$, $\mathbb{R}^2$ has basis $\{(1, 1), (1, 0)\}$.

3. Consider the vector space $\mathbb{R}^3$. Determine whether or not S is a subspace, for $S = \{(a, b, c) : a + b = c\}$.

   Need to check closure under vector addition and scalar multiplication.
   VA: $(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$. We assume that $a_1+b_1 = c_1$ and that $a_2+b_2 = c_2$. Adding these we get $a_1+b_1+a_2+b_2 = c_1+c_2$. Rearranging: $(a_1 + a_2) + (b_1 + b_2) = (c_1 + c_2)$, so S is closed under VA.
   SM: $d(a, b, c) = (da, db, dc)$. We assume that $a + b = c$, multiplying by d we get $da + db = dc$. Hence S is closed under SM, and is a subspace.

4. Consider the vector space $\mathbb{R}^2$. Show that the following set is dependent: $\{(1, 2), (3, 4), (5, 6)\}$.

   Solution 1: The dimension of $\mathbb{R}^2$ is 2, which is the maximal size of an independent set. This set must therefore be dependent.
   Solution 2: $1(1, 2)-2(3, 4)+(5, 6) = (0, 0)$ is a nondegenerate linear combination of these vectors yielding $(0, 0)$. Other linear combinations are possible.

5. Consider the vector space $\mathbb{R}^2$. Show that the following set is spanning: $\{(1, 2), (3, 4), (5, 6)\}$.

   Given any (x, y) in $\mathbb{R}^2$, we need to find some a, b, c so that $a(1, 2) + b(3, 4) + c(5, 6) = (x, y)$.
   Many solutions are possible; for example $a = -2x + 1.5y, b = x - 0.5y, c = 0$. Observe that $(-2x+1.5y)(1, 2)+(x-0.5y)(3, 4)+0(5, 6) = (-2x+1.5y, -4x+3y) + (3x - 1.5y, 4x - 2y) + (0, 0) = (x, y)$.

   Note: It is not correct to claim that this set is spanning because it contains three vectors and $\mathbb{R}^2$ has dimension 2. For example, $\{(1, 0), (2, 0), (3, 0)\}$ contains three vectors, but is not spanning.